A RENOVATION OF THE EINSTEIN SEDIMENT FUNCTION

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Abstract: calibrated and verified method for computing bed material discharge rates in open channel flows is presented. It has two main components: a Revised Sediment Function (RSF) and a Sediment Function Spreadsheet Application Model (SFSAM). The (RSF) is a revision of the Einstein (1950) Sediment Function (ESF) to improve the accuracy of its computed results and the (SFSAM) is a solution to the complexity and time consuming application of the (ESF) method. Sample applications and comparisons with other methods results are presented.

INTRODUCTION

Problem definition: The search for practical and reliable methods to describe and quantify the process of interaction between fluid flow and sediment has been a great challenge for engineers and scientists. The Einstein Bed-Load Function for Sediment transportation in open Channel flows (1950), referred to in this paper as the Einstein Sediment Function (ESF) is a considerable achievement in this regard. Over the years, the (ESF) has had its good share of praises, reviews, criticisms, and concerns. Some of the concerns are: (1) the method tends to over-estimate the sediment discharge rate as reported by Simon and Senturk (2001) and Stall, et al (1958); (2) the application of the method is complicated and time consuming; and (3) it requires the use of many complex graphs and equations.

Scope and objectives: The main objective of this paper is to assess the above three concerns regarding the (ESF) method, develop necessary means of improving the accuracy and reliability of its computed results and increase the efficiency of its application.

OVERVIEW OF THE EINSTEIN SEDIMENT FUNCTION

Basic concept: The (ESF) is based on good science, sensible engineering, some assumptions and results of experimental observations and studies. Its computation of the rate of sediment discharge is based on the basic concept that the probability (P) of the fluid hydrodynamic lift force (L), acting on a representative sediment particle of grain size (D), exceeds its submerged weight (W_b). Following is a concise summary of the (ESF) development.

Statistical distribution of the lift force (L): Einstein and Elsamni (1949) found that (L) has two parts: an average part ($\bar{L}$) and a turbulent fluctuating part $(L^{'})$, and behaves like a random variable with mean$(\bar{L})$ and standard deviation ($\sigma = \bar{L}\eta_0$), where $\eta_0 = 0.5$ and $(\bar{L})$ can be expressed as:

$$\bar{L} = (C_1)(C_L)(\frac{\gamma_f}{g})(D^2)(U_*^{'2})(log_{10}^2 10.6\left(\frac{X}{\Delta}\right)) \tag{1}$$

In which (C_1) is a constant, (C_L) is a lift force coefficient = 0.178, (γ_f) is the fluid density, (g) is the acceleration of gravity, $(U_*^{'})$ is the shear velocity with respect to the grains and(X), (Δ) are roughness parameters as defined by Einstein (1950).

<u>The probability (P)</u>: The probability (P) is described as the probability that, $(\frac{W_b}{L})$ <1. Einstein (1950), using a detailed and somewhat complex formulation, concluded that (P) can be written as:

$$\mathrm{P} = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_* \psi_* - \frac{1}{\eta_0}}^{B_* \psi_* - \frac{1}{\eta_0}} e^{t^2} dt \tag{2}$$

In which (B_*) is a constant =0.143, (t) is a variable of integration and (ψ_*) is defined as:

$$\psi_* = \text{flow intensity} = \left(\frac{\gamma_s - \gamma_f}{\gamma_f}\right)\left(\frac{gD}{U_*^{'2}}\right) \xi \, Y (\frac{\beta}{\beta_x})^2 \tag{3}$$

Where (ξ), (Y) and $(\frac{\beta}{\beta_x})^2$ are, respectively, correction factors for: particle hiding in the laminar sub layer (∂), the lift coefficient (C_L) and the logarithmic velocity distribution due to bed material -mixture, as given by Einstein (1950).

<u>The exchange of particles between bed and motion:</u> Another expression for (P) was obtained by considering the stability of the bed and the equilibrium of exchange of particles between the bed and the flow and is given as:

$$P = \frac{A_* \emptyset_*}{1 + A \emptyset_*} \tag{4}$$

Where (A_*) is another constant to be determined by calibration using measured data and (Φ_*) is the intensity of bed-load transport, and defined as:

$$\emptyset_* = \left(\frac{q_B}{\gamma_s}\right)(\frac{\gamma_f}{\gamma_s - \gamma_{f_s}})^{\frac{1}{2}} (\frac{1}{gD^3})^{\frac{1}{2}} \tag{5}$$

In which (q_B) is the bed load discharge rate per unit channel width.

<u>The Einstein Sediment Function (ESF):</u> Combination of equations ((2) and (4) produces the (ESF), given as:

$$P = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_* \psi_* - \frac{1}{\eta_0}}^{B_* \psi_* - \frac{1}{\eta_0}} e^{-t^2}\, dt = \frac{A_* \phi_*}{A_* \phi_* + 1} \qquad (6)$$

Application of equation (6), to the computation of (q_B) is very elaborate, time consuming and uses many equations and graphs, as demonstrated by Einstein (1950).

RENOVATION OF THE EINSTEIN SEDIMENT FUUNCTION (ESF)

The renovation of the (ESF) followed two main tracks: the objective of the first is to identify possible sources of the over-estimating concern and the second is to find ways of simplifying and expediting the computation process. The first objective is achieved by revising the probability (P) of equation (2) and the second by the development of the Sediment Function Spreadsheet Application Model (SFSAM), as described below.

Revision of the probability (P): The derivation of the probability (P) of equation (2) was revisited using an approach similar to the one used by Kadib (1966) on his work covering sand transport by wind. Accordingly, equation (2) can be written as:

$$P = Pr\,(L > W_b) = Pr\,(\bar{L} + L' > W_b) = Pr\,(L' > W_b - \bar{L}) = Pr\,\left(\frac{L'}{\eta_0 \bar{L}} > \frac{W_b}{\eta_0 \bar{L}} - \frac{1}{\eta_0}\right) \qquad (7)$$

By using the logarithmic formula for the velocity distribution over a bed of uniform sediment and at a distance of .35D above the bed, the ratio ($\frac{W_b}{\eta_0 \bar{L}}$) of equation (7) can be written as ($\frac{W_b}{\eta_0 \bar{L}}$) = ($B_* \psi_*$), where ($\psi_*$) is the flow intensity as defined by equation (3) and (B_*) is a constant. Now, and since ($\frac{L'}{\eta_0 \bar{L}}$) is a new random variable with mean (μ) = zero and standard deviation (σ) =1, the probability (P) of equation (7) can be expressed as:

$$P = \frac{1}{\sqrt{2\pi}} \int_{B_* \psi_* - \frac{1}{\eta_0}}^{\infty} e^{-t^2/2}\, dt \qquad (8)$$

<u>The Revised Sediment Function (RSF):</u> The combination of equations (4) and (8) produces the Revised Sediment Function (RSF) which reads:

$$P = \frac{1}{\sqrt{2\pi}} \int_{B_*\psi_* - \frac{1}{\eta_0}}^{\infty} e^{-t^2/2}\, dt \quad = \frac{A_*\phi_*}{1+A_*\phi_*} \tag{9}$$

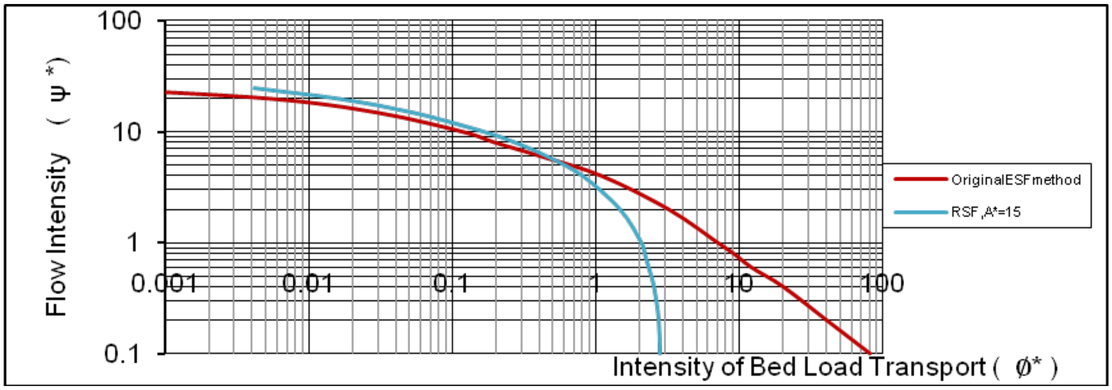

Figure 1. Effect of the revised probability (P) on the (ESF)

Figure 1 is a comparison between the computed (ϕ_*) values using equation (9) of the (RSF) method with those of the (ESF). It can be seen from the results of Figure 1 that: (1) for low flow and sediment transport conditions where the probability (P) is small (ψ_*>5, there is a fair agreement between the results of the two methods; (2) For higher flow conditions where, (P), is relatively high, (ψ_*<5), the (ESF) estimates are much higher than the (RSF) results with increasing factors as (ψ_*) values decrease{~200% for (ψ_*) =2 and ~500% for (ψ_*) = .6}. This is similar to the trend reported by Simon and Senturk (2001) on the comparisons between the (ESF) computed results with those of Colby (1964) method for Big Sand Creek and Stall et al (1958) for Money Creek .The (ESF) results were higher by factors of 400% for Big Sand Creek and 225% for Money Creek. This is a clear indication that replacing equation (2) of the (ESF) with equation (8) of the (RSF) has solved the over-estimating concern of the (ESF) results.

<u>The Sediment Function Spreadsheet Application Model (SFSAM):</u> The (SFSAM) is a solution to the complex and time consuming characteristics of the (ESF). It is based on equation (8), with (B_*)=.143, (η_0) = .5 and (A_*)=15, and uses the computation steps of Einstein(1950) sample example of Big Sand Creek, with one exception regarding the computation of the suspended load .The (SFSAM) computes the suspended load separately by the summation of its incremental components over the channel depth.Table1shows an example of the in-put/out-put parameters of (SFSAM), where the in-puts are: (*ib*) fraction of bed material in size (*D*), (*R'*) hydraulic radius due to grain roughness, (S) the bed slope, (D_{35}) and (D_{65}) grain sizes of which 35% and 65% are finer respectively and (*g*), (γ_s), (γ_f), are as defined before. The outputs are: the channel discharge per unit width (q_w) and ($i_B q_B$), ($i_B q_S$), and ($i_B q_T$) which are respectively, the bed load, suspended load and total bed material load, per unit width for the given grain size (D).The (SFSAM) could be made available to interested parties, if requested.

Table1. Example of (SFSAM) in-put/out-put parameters

$D(mm)$	0.495	q_w (cfs/ft)	27.296	S	0.00105	γ_f (lb/ft^3)	62.4
$D(ft)$	0.001624	$i_B q_S$ (lb/f/s)	0.082733	D_{65} (ft)	0.001148	γ_s (lb/ft^3)	165
ib	0.178	$i_B q_B$ (lb/f/s)	0.034856	D_{35} (ft)	0.000951	D_{35} (mm)	0.29
$R'(ft)$	3	$i_B q_T$ (lb/f/s)	0.1176	g(ft/s^2)	32.2	D_{65} (mm)	0.35

THE (RSF) VERIFICATION AND CALIBRATION

Verification data: The reliability and accuracy of equation (8) and the (RSF) are checked and verified by comparing their computed results with available measured values for natural streams, as well as, computed rates by a number of well- known formulas. Three well documented verification sets are used. The first two sets include measured sediment rates for Niobrara River near Cody Neb. and the Colorado River at Taylors Ferry. They cover data obtained from field measurements and computation by several known formulas, as reported by Vanoni (1975) and shown in Figure 2 where the (RSF) computed results are also plotted. The third set includes a comparison of computed sediment discharge rate using Einstein (1950) and Colby (1964) methods for Big Sand Creek. Miss., as reported by Simon and Senturk (2001) and shown in Figure 3 where the (RSF) computed results are also shown.

Verification results: The comparisons presented in Figure 2 show that the ((RSF) method provides the closest and the best estimate of the total sediment discharge in both Niobrara and Colorado Rivers. The results of Big Sand Creek comparison, shown in figure 3, clearly indicate that the (RSF) estimates are more consistent with Colby methods than the (ESF). Again, these results prove the validity of the (RSF) and equation (8) in correcting the critical concerns regarding the accuracy of the (ESF) method.

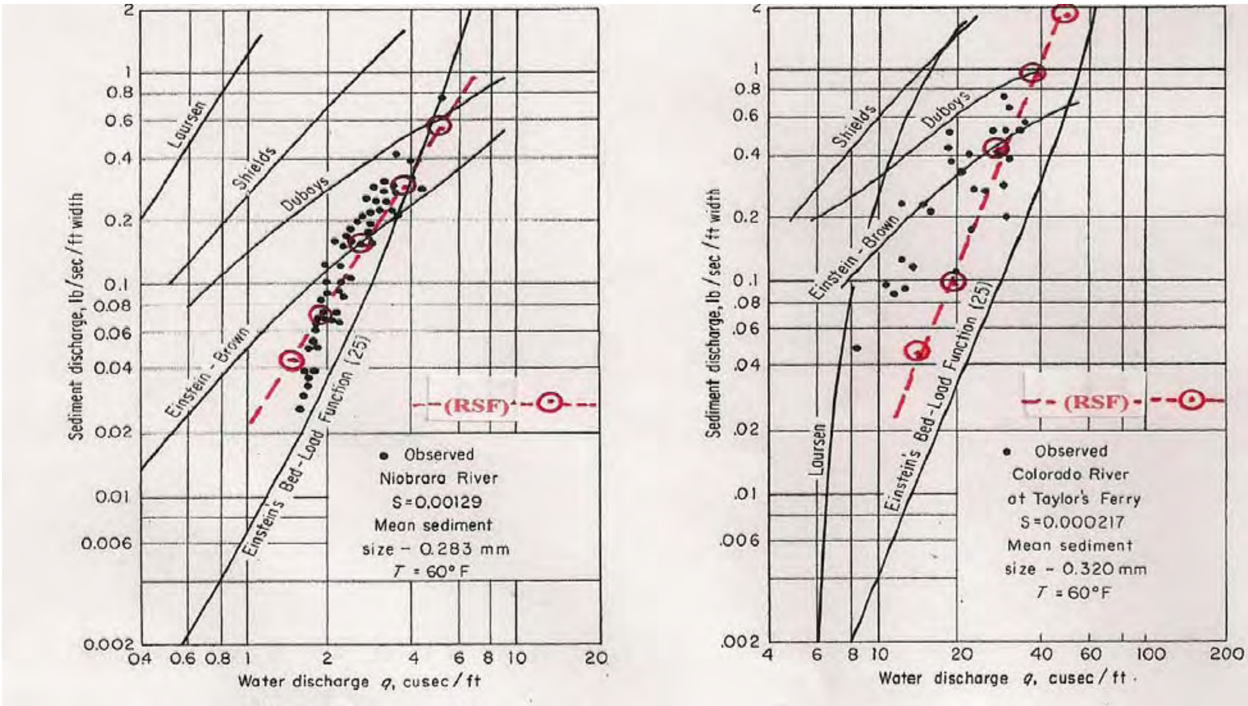

Figure 2. (RSF) Verification using measured and computed data (Niobrara and Colorado Rivers)

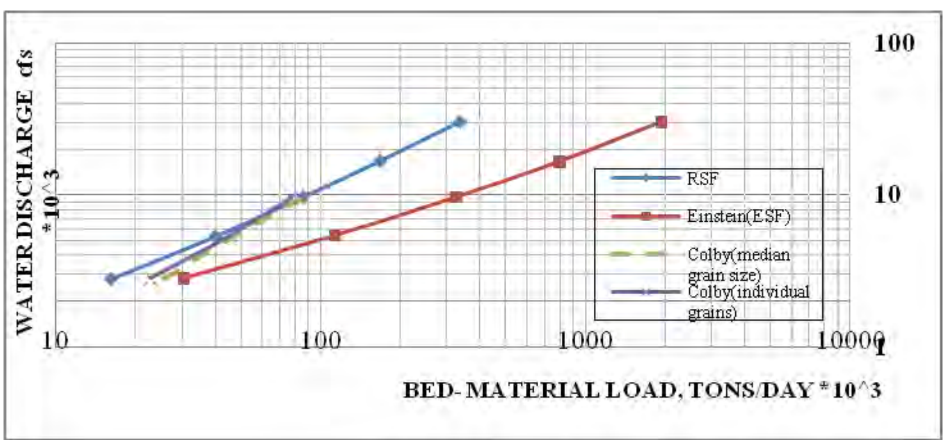

Figure 3. Comparisons between (RSF), (ESF) and Colby method, Big Sand Creek, Miss.

PRACTICAL APPLICATION OF THE (RSF) METHOD

Application of the (RSF) procedures to a particular channel reach assumes wide channel and computes the bed load, suspended load and total load, per unit width for a given grain size fraction.

Table 2. (SFSAM) Model Run, Niobrara River

The Sediment Function Spreadsheet Application Model (SFSAM)							
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River	***Niobrara***						
Q(cfs)	**4.6**				**lb/ft/s**		**0.17**
Width(ft)	**1**	**bed material discharge per unit width for a given grain size fraction**					
					Tons/day		**7.37**
qw(cfs/ft)	***4.60***		**(qw)/(qw)m**		**1.00**		
D(mm)	***0.25***	$(q_w)_m$***(cfs/ft)***	***4.62***	***S***	***0.00129***	γ_f**(lb/ft^3)**	***62.4***
D(ft)	***0.00082***	$i_B q_S$***(lb/f/s)***	***0.15303***	D_{65}***(ft)***	***0.0011***	γ_s***(lb/ft^3)***	***162***
ib	***0.38***	$i_B q_B$***(ib/f/s)***	***0.0175***	D_{35}***(ft)***	***0.00076***	D_{35}***(mm)***	***0.233***
R'(ft)	***0.54***	$i_B q_T$***(lb/f/s)=***	***0.1705***	***g(ft/s***^{2}***)***	***32.2***	D_{65}***(mm)***	***0.335***

Table 3. Total bed material transport rates using the (RSF) method, Niobrara River

	River name:		*Niobrara. Near Cody, Neb.*					
	Slope:		*0.00129*					
	Width (ft)	*1*	*1*	*1*	*1*	*1*		
Total Bed material load per unit width for a given grain size($i_B q_T$),in Ib/f/s								-
D(mm),ib	q_w (cfs)/ft	1.5	2	3	4	5		*4.6*
.55, .2		0.0252	*0.0273*	*0.0339*	0.0415	0.0492		*0.0463*
.35, .28		*0.0116*	*0.0171*	*0.0339*	*0.0596*	*0.0803*		*0.0762*
.25, .38		*0.0052*	*0.0193*	*0.0506*	*0.1117*	*0.2168*		***0.1705***
.175, .1		*0.0011*	*0.0054*	*0.0458*	*0.1102*	*0.2209*		*0.1724*
$i_T q_T$*(b/ft/s)=* $\Sigma(i_B q_T)$		*0.0431*	*0.0691*	*0.1642*	*0.323*	*0.57*		*0.4654*

Computation steps: Practical application of the (RSF) is easy, fast and very friendly and covers four main steps as outlined below using Niobrara River data as an example.

Step (1): Enter the (SFSAM) with the given sediment and flow parameters as shown in Table 2 and a value of (*R'*) that makes : $(q_w) / (q_w)m = 1$.

Step (2): Read the instantly computed value of bed material discharge rate ($i_B q_T$) for the selected size fraction

Step (3): Repeat step (2) for each of the remaining (D,i_b) combination.

Step (4): The total bed material transport per unit width for all grain size fractions, ($i_T q_T$) =

($q_{sediment}$), is the summation of steps (2) and (3), as shown in Table3.

Application to Yang's Example: Another practical demonstration of the application of the (RSF) method is the example given by Yang (1996) where he presented the computed results of sediment transport rate for the Niobrara River using six well known formulas and a comparison of these results with the measured bed material discharge. Yang's analysis assumes the following conditions: bed slope (S) = .00144, water discharge (q_w) =4.6 (ft^3/s)/ft, bed material is fairly uniform with $(D_{50}) = .283\ mm$ and field measured bed material discharge rate (*iTqT)* = ($q_{measured\ sediment}$) =0.46 (lb/s)/ft. The present task is to conduct the same comparison of Yang's example, using the (RSF) method under the following four scenarios:

Scenario 1: Compute ($q_{sediment}$) using the same conditions of Yang's example,

Scenario 2: Compute ($q_{sediment}$) using the grain size fractions given in Table 3 for the Niobrara River and all other conditions as given in Yang's example,

Scenario 3: Compute ($q_{sediment}$) as scenario 2, except for S = .00129 as reported by Vanoni (1975).

Scenario 4: Compute ($q_{sediment}$) as scenario 3 and assuming uniform bed material.

Results of the analysis using the (RSF) method: The four scenarios common in-puts into the (SFSAM) are: (q_w) = 4.6 (ft^3/s)/ft, an (R')that makes $(q_w)_m$=(q_w) =4.6 ft^3/s)/ft, (D_{35})=.233 mm, (D_{65})= .335mm, (γ_s) = 162 lb/ft^3), (γ_f) = 62.4 lb/ft^3) and g = 32.2 ft/s^2.

The computed results, using (SFSAM) are:

Scenario 1: In-puts are: D= .283 mm, i_b = 1 (uniform grain size), and S = .00144

Results: ($q_{sediment}$) = 0.4376= ($i_T q_T$), (as shown in Table 4).

Scenario 2: In-puts are: (D,i_b) of (.55, .2), (.35, .28), (.25, .385) and (.175, .1) as shown in table 3 and S=.00144

Results: ($q_{sediment}$) =$\sum(i_B q_{T})$ =0.6003= ($i_T q_T$), (as shown in Table 5).

Table 4. Results of (SFSAM) run, Scenario 1

D(mm)	0.283	$(q_w)_m(cfs/ft)$	4.59	S	0.00144	$\gamma_f (lb/ft^3)$	62.4
D(ft)	0.00093	$(i_T q_S)$ (lb/f/s)	0.36032	D_{65}(ft)	0.0011	$\gamma_s (lb/ft^3)$	162
ib	1	$(i_T q_B)$ (lb/f/s)	0.0773	D_{35} (ft)	0.00076	D_{35} (mm)	0.233
R'(ft)	0.54	$(i_T q_T)$ (lb/f/s)	0.4376	g(ft/s^2)	32.2	D_{65}(mm)	0.335

Table 5. results of (SFSA) run, Scenario 2

D(mm), ib		.55, .2	.35,.28	.25, .385	.175,.1
$i_B \boldsymbol{q_T}$*(lb/f/)s*		0.0504	0.0800	0.2369	0.2330
	q(sediment) (lb//f)/s $=\Sigma i_B q_T$ = 0.6003				

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Table 6. Results of (SFSAM) run, Scenario 4

D(mm)	0.283	$(q_w)_m(cfs/ft)$	4.60	Se	0.00129	$\gamma_f (lb/ft^3)$	62.4
D(ft)	0.00093	$i_T q_S$(lb/f/s)	0.25887	D65(ft)	0.0011	$\gamma_s (lb/ft^3)$	162
i_b	1	$i_T q_B$ (lb/f/s)	0.06353	D35(ft)	0.00076	D_{35}(mm)	0.233
R'(ft)	0.538	$i_T q_T$ (lb/f/s)=	0.3224	g(ft/s^2)	32.2	D_{65}(mm)	0.335

Scenario 3: In-puts are: the same as scenario 2, S = .00129 as reported by Vanoni (1975)

Results: $(q_{sediment})$ = .4654= $(i_T q_T)$,(as shown in Table 3)

Scenario 4: In-puts are: D= .283 mm, ib = 1(uniform grain size), S=.00129

Results: $(q_{sediment})$ = 0.3224= $(i_T q_T)$, (as shown in Table 6).

<u>Comparison of (RSF) results with Yang's example:</u> Table 7 summarizes the results of the computed total bed material transport rate ($q_{sediment}$) for the Niobrara River using Yang's example and the four given scenarios. Comparison of these results with the measured rate ($q_{measured\ sediment}$) = 0.46 (lb/s)/ ft is also shown in Table 7 as the ratio $q_r = (q_{sediment})/(q_{measured\ sediment})$. Based on these analysis and comparisons, the following conclusions regarding the performance of the (RSF) are made:

Table 7. Summary of comparison of (RSF) results method with measured and other formulas

Data source	**Formula or method**	($q_{sediment}$) **(lb/s)/ft**	q_r
Yang (1996)	Du Boys	1.06	2.3
Yang (1996)	Shields	5.46	11.87
Yang (1996)	Schoklitsch	0.193	.42
Yang (1996)	Meyer-Peter	0.059	.13
Yang (1996)	Meyer-Peter and Müller	0.026	.057
Yang (1996)	Rottener	0.293	.64
Scenario 1	(RSF), uniform grain size and S=.00144	0.438	.95
Scenario 2	(RSF), grain fractions, S=.00144	0.6	1.3
Scenario 3	(RSF), grain fractions, S=.00129	0.466	1.01
Scenario 4	(RSF), uniform grain Size and S=.00129	0.3224	0.7

(1) The (SFSAM) is a fast and efficient method for processing the bed material transport rate computations in open channel flows.

(2)The (RSF) provides the best agreement with field measurements and its results are accurate and reliable.

(3) As indicated by the results of scenarios 3 and 4 , the use of individual size fractions (i_b) and its selected representative grain size (D), gives more accurate results than assuming uniform bed material grain size.

(4)The channel bed slope (S) is an important parameter in both the computation and verification processes. In the application of the Niobrara River verification data set, Vanoni (1975) reported a slope of .00129, while Yang (1996) selected a 10% steeper slope of .00144, which resulted in approximately 30 % increase in the computed sediment rate (scenarios 2 and 3 of Table 7).

SUMMARY AND CONCLUSIONS

A calibrated and verified method for the computation of the total bed material transport rate in open channel flows is developed and presented. The method has two main components: A Revised Sediment Function (RSF) and a Sediment Function Spreadsheet Application Model (SFSAM). The (RSF) is based on the Einstein Sediment Function (ESF), and a revised derivation of its probability equation. The (SFSAM) is an efficient and fast vehicle that conducts all the necessary hydraulic and sediment computations of the (RSF). The main findings of this paper are summarized as:

(1) A major source of the over-estimating concern, of the (ESF) computed sediment transport rate, is identified as an apparent error in the derivation and application of the probability (P) of equation (2) of the Einstein Sediment Function.

(2) The above concern is corrected by developing and applying a revised version of the probability (P) of equation (2) and replacing it by equation (8).

(3) Concerns and issues regarding the complexity and time consuming characteristics of the (ESF) method are resolved by the development and application of the (SFSAM), a fast computing model that conducts all the hydraulic and sediment computations of the (RSF).

(4) The (RSF) and the (SFSAM) model results are calibrated and verified using two sets of measured and computed field sediment transport data for the Niobrara River near Cody, Nebraska and Colorado River near Taylors Ferry. Additional calibration set covers the classic case of sediment rate computations for Big Sand Creek, Mississippi.

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(5) The (RSF) and its companion, the (SFSAM), present a long- awaited renovation of the (ESF) that make its results accurate and reliable and its application an enjoyable and pleasant exercise.

ACKNOWLEDGEMENTS

This paper is dedicated to the memory of Dr. Hans Albert Einstein who educated and introduced me to the subject of sediment transport. My graduate students at California State University Long Beach motivated me to take this adventure by their curiosity and stimulating questions about the Einstein Bed-Load Function. My wife Nadia encouraged me to work on this paper and gave me continuous support. Many thanks are due to my granddaughter Jacqueline Camarena for her help in editing the manuscript.

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