SHORTCOMINGS OF TWO-PARAMETER POWER FUNCTIONS FOR FITTING BEDLOAD RATING CURVES

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INTRODUCTION

The capacity to estimate the sediment loads of streams is a key component of numerous types of geomorphic investigations (Andrews, 1986; Inman and Jenkins, 1999; Singer and Dunne, 2001; Syvitski et al., 2005; Klonsky and Vogel, 2011). Sediment loads at defined locations along a stream are frequently calculated with empirical sediment rating curves developed by statistically fitting the parameters of a transport model to a set of paired water discharge and sediment transport measurements.

Various transport models can be fit to sediment transport and stream flow data. Whether a particular model is appropriate depends on the purpose of the rating curve. Often, the purpose of fitting a rating relation is simply to estimate sediment loads. In cases where transport computations are confined to interpolated values within the range of observed discharges, the form of the fitted equation and the values of the fitted parameters may be of little importance. However, the use of incorrect functional models can lead to large errors when extrapolation beyond the observed range of discharges is required (Glysson, 1987). Moreover, sediment rating curve parameters are also sometimes used to investigate the transport process by inferring relationships between the fitted parameters and physical attributes of the system (Asselman, 2000; Barry et al., 2004; Gaeuman, 2010). Where extrapolation is required or when parameter values are to be assigned physical meaning, the use of appropriate fitting procedures and a functional form that accurately represents the transport process are critical.

Perhaps the most common transport model used for this purpose is a simple 2-parameter power function:

$$Q_s = aQ^b \qquad (1)$$

where Q_s is the sediment transport rate, Q is the water discharge, and a and b are parameters. This method may serve as a reasonable approximation for the relationship between flow and suspended sediment transport, but it provides an unrealistic representation of coarse bedload transport. The existence of a threshold for sediment entrainment below which coarse sediment is immobile is a fundamental concept in bedload transport theory. Such a threshold can be accommodated by fitting bedload transport data with a shifted power function in which a discharge threshold required for the initiation of sediment transport (Q_c) is subtracted from Q:

$$\begin{aligned} Q_s &= a(Q - Q_c)^b \qquad for\ Q_c > Q \qquad (2) \\ Q_s &= 0 \qquad otherwise \end{aligned}$$

This 3-parameter transport model is analogous to numerous sediment transport equations (e.g. Meyer-Peter and Müller 1948) that express the sediment transport rate as a function of the excess

dimensionless shear stress exceeding a critical value defined by the Shields number. Inclusion of Q_c in the rating relation accounts for the behavior of coarse bedload fractions that remain motionless at small discharge levels. Where the data show that no such threshold exists, Q_c can take a value of zero, thereby reducing (2) to a standard two-parameter power function.

Although equation (2) offers a more flexible and realistic representation of the bedload transport process, a large proportion of investigators prefer to base their analyses on the simpler two-parameter model. Given the popularity of the simple power function for representing bedload transport, it is prudent to consider the effect arbitrarily forcing Q_c to zero has on the quality of the fitted rating curve. This paper presents a numerical experiment demonstrating that incorrectly ignoring non-zero values of Q_c can produce gross errors in the fitted values of both *a* and *b*. The significance of those errors for various rating curve applications is discussed and demonstrated using bedload measurements collected in the field.

EXAMPLE USING SYNTHETIC DATA

The potential impact of inappropriately imposing a two-parameter power function on a system where a non-zero entrainment threshold exists is demonstrated by fitting synthetic data drawn from a hypothetical rating curve defined by equation (2) with *a*, *b*, and Q_c set to 0.1, 2, and 1.5 m^3/s. Eleven paired Q-Q_s data points spanning a range of Q from 1.53 to 3 m^3/s were drawn from the curve. No error was added to the computed values of Q_s, so the 11 data points matched their parent curve exactly. Those data were then fit with a 2-parameter power function given by equation (1) using the linear regression functionality in Microsoft Excel, which implements linear least squares with log transformed data. As illustrated in Figure 1, that procedure results in a poor fit that deviates markedly from the actual parent curve used to generate the data. The value of the fitted exponent *b* of 9.31 is 4.66 times larger than the actual value of 2, and the value of the fitted coefficient is 5 orders of magnitude too small. These results are typical – in general, incorrectly forcing Q_c to zero invariably results in overestimation of *b* and, for datasets in which Q is predominantly greater than unity, underestimation of *a*.

The magnitude of the errors depends to a large extent on how much greater the values of Q in the dataset are than Q_c. For the example of Figure 1, the data used to fit the power function spans a range of Q between 1.02 times Q_c and $2Q_c$. If the range of Q used to fit the curve is limited to $1.02Q_c$ to $1.9Q_c$ while holding all other conditions constant, the estimated value of *b* increases to 9.86, or 4.93 times the correct value, and the estimated value of *a* is further reduced to 0.000013.

These errors in the parameter estimates can produce substantial errors in sediment loads computed with equation (1), although the nature and magnitude of the errors will depend on the characteristics of the hydrograph the rating curve is applied to. It is clear from Figure 1 that the 2-parameter fit to the data will under-predict transport rates associated with discharges near the center of the data range (e.g., the transport rate computed for 2 m^3/s with the 2-parameter curve is just 44 percent of the correct value). The potential for overestimation of transport rate at discharges beyond the range of the sample data, however, is a more significant problem. Although sediment rating curves should, in principle, be based on sample data spanning the full range of the flows they will be applied to, that is not always possible. Sampling at the highest discharges may be impractical for numerous reasons, including safety considerations, equipment limitations, lack of funds, or time constraints. Large flow events often remain unsampled, such

that load estimates for those events must be based on measurements obtained at relatively small discharges.

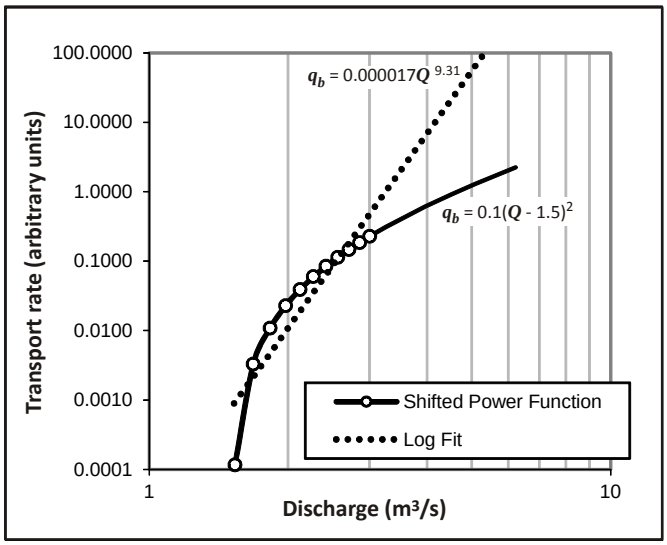

Figure 1 Comparison between a shifted power function [equation (2)] with known parameters and a fitted curve based on a 2-parameter power function [equation (1)] with parameters fit by ordinary least squares. The data points drawn from the shifted power function used in the fitting operations are indicated by the open circles.

The magnitude of the potential errors caused by extrapolating a 2-parameter rating curve beyond the range of the data is illustrated using the synthetic hydrograph shown in Figure 2. That hydrograph starts at 1.5 m^3/s (equal to Q_c) and rises at a steady rate of 0.15 m^3/s per day for 20 days to a peak of 4.5 m^3/s ($3Q_c$). The peak is maintained for 3 days, after which flow recedes at an exponential rate of 1.5 percent per day for 71 days.

At $3Q_c$ and 3 days, neither the magnitude nor the duration of the peak is extraordinary, and the mean discharge over the full hydrograph (2.8 m^3/s) is within the range of the transport data. Nonetheless, greatly overestimated transport rates at the higher discharges produces large errors in the total computed load. The 2-parameter rating curve based on data spanning 1 to 2 times Q_c (Figure 1) produces an estimated transport rate for 4.5 m^3/s that is 23 times larger than the correct value and a total load for the full hydrograph that is 10.2 times too large. If the 2-parameter rating curve based on transport data spanning 1 to 1.9 times Q_c is used, the maximum transport rate is 40 times too large and the total load is overestimated by a factor of 17.

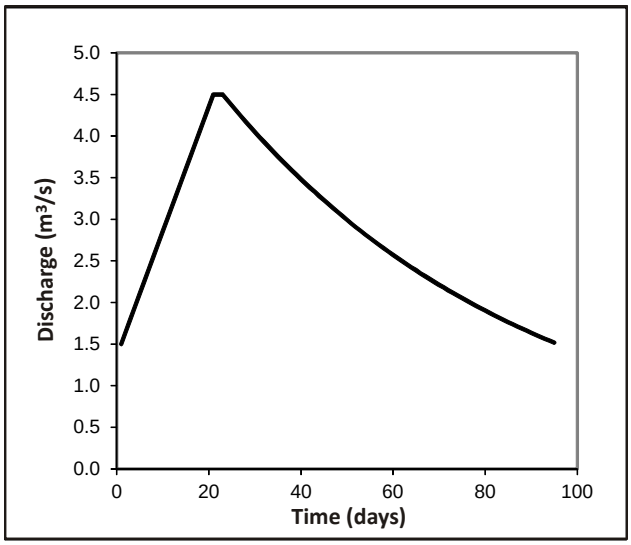

Figure 2 Synthetic hydrograph used to demonstrate the effect of extrapolating a 2-parameter rating curve to higher discharges when estimating total sediment loads.

EXAMPLE USING FIELD DATA

The data presented in this section consists of 18 bedload transport samples collected in Rush Creek, a tributary to the Trinity River in northern California, in water years 2004 and 2005 (Figure 3). These bedload samples were obtained at discharges ranging from 3.4 to 13.4 m^3/s, which slightly exceeds the stream's mean annual daily maximum flow of 11.6 m^3/s, based on 10 years of available record (water years 2004-2013, USGS 11525530 Rush Creek near Lewiston, CA). The annual mean flow in the creek over the same 10-year period is 1.2 m^3/s.

When fitted with a rating curve of the form given by equation (2), these data yielded estimates of a, b, and Q_c of 1.243, 2.415, and 1.8 m^3/s, respectively, whereas fitting these same measurements with a 2-parameter power function produced estimates of a and b of 0.0798 and 3.422. As with the previous example, forcing Q_c to equal zero by imposing equation (1) substantially increased the estimated value of b (by 42 percent in this example) and decreased the estimated value of a by orders of magnitude.

Elevations in the Rush Creek watershed range from about 540 m above sea level at the confluence with the Trinity River to more than 2200 m at the creek's headwaters in the Trinity Alps. Consequently, relatively long-duration flood events can occur during the spring snowmelt, but the largest floods tend to be brief events associated with intense winter storms that produce heavy rain in the lower elevations and rain-on-snow higher in the mountains. Acquiring sediment transport samples over the full range of flows therefore requires the ability to quickly mobilize a sampling crew during the relatively sudden and short-duration peaks that occur during winter storms.

In the case of the 2004 and 2005 Rush Creek data presented here, neither the instantaneous peak discharge nor a discharge equaling the maximum daily mean were sampled. Both maxima were attained during a storm on February 17, 2004 when the instantaneous peak flow reached 54.9

m^3/s and the daily mean flow was determined to be 25.7 m^3/s. Although an attempt was made to collect bedload samples during that storm, none were obtained until the following day (February 18) when discharge was between 13.4 and 10.7 m^3/s (Figure 4). Thus, computing the total sediment load for any time period that includes water year 2004 requires extrapolation beyond the range of the measured data.

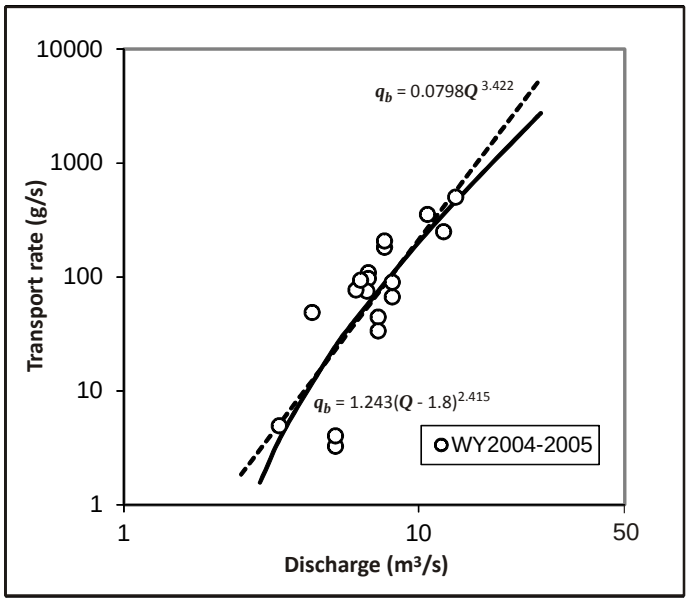

Figure 3 Comparison between a 3-parameter rating relation incorporating an entrainment threshold and a 2-parameter power function fit to bedload data collected in Rush Creek.

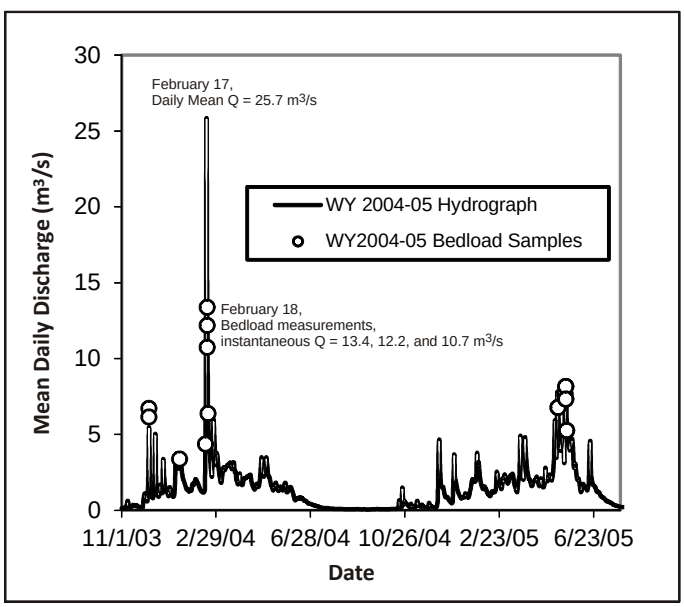

Figure 4 Daily mean discharge in Rush Creek in water years 2004 and 2005. Open circles indicate days on which bedload samples were acquired and the corresponding discharges.

Application of the 3-parameter rating relation based on equation (2) to the daily mean flows recorded for Rush Creek in water year 2004 results in a total bed sediment load of 305 metric tons, whereas application of the 2-parameter curve fit the same data yield a total load of 560 metric tons. Thus, omission of Q_c from the rating relations developed from Rush Creek bedload samples increases the total estimated bed sediment load for the year by a factor of more than 1.8. The increase would be even larger if the instantaneous peak, which was more than twice as large as the daily mean, had been considered in the analysis. It is also worth pointing out that the nearly 2-fold overestimation of the load described here arises from a hydrological record containing just one day on which the daily mean flow exceeded the range of measurements. It seems reasonable to suppose that, in some instances, it may be necessary to compute loads from hydrological records that exceed the range of sediment transport measurements on multiple days.

IMPLICATIONS FOR INTERPRETING TRANSPORT PROCESSES

Several previous authors have attempted to link variability in bedload rating curve parameters to differences in site conditions and the physical processes involved in bedload transport (Emmett and Wolman, 2001; Barry et al., 2004; Bunte et al., 2006). According to Barry et al. (2004), variations in b at different stream locations are inversely proportional to q^*, the measure of relative substrate armoring introduced by Dietrich et al. (1989). Similarly, Bunte et al. (2006) reported an inverse correlation of b with armoring as expressed by the ratio of the surface to subsurface median particle sizes, as well as a positive correlation with stream width.

Given that both of these studies used 2-parameter rating curves, it seems likely that the reported variations in b are related to differences in the relative magnitudes of Q_c at the various sampling locations as much as any other factor. That is, use of a 2-parameter rating curve should be expected to produce larger values of the exponent b for sample data sets associated with smaller values of Q_U/Q_c, where Q_U represents the upper limit of the range of the discharges sampled. The reported correlations between b and measures of substrate armoring and channel geometry may arise simply because those measures co-vary with Q_c. This interpretation is consistent with the observations of Emmett and Wolman (2001), who noted a positive correlation between b and the median and 90th-percentile bed surface particle sizes and suggested that delayed onset of bedload mobility (i.e., a larger Q_c) due to the presence of a coarse armor layer on the streambed causes the rating relation to steepen (i.e., increase in b). If correct, the interpretation offered here implies that the value of b obtained from equation (1) may depend on attributes of the sampling effort, such as the range of discharges over which bedload samples were obtained, as much as any physical attributes of the stream.

Statistical relationships have also been reported between values of the coefficient a and certain physical parameters. Barry et al. (2004) observed an inverse correlation between a and drainage basin area, and attributed it to a proportionality between a and the ratio of sediment transport rate and stream discharge. Of five statistical relationships between a and physical parameters related to stream substrate or geometry reported by Bunte et al. (2006), three (bankfull width, basin area, and bankfull discharge) involve inverse relationships with alternative measures of stream scale. Similarly, Syvitski et al. (2000) found an inverse correlation between a and long-term mean discharge, although that study considered fine suspended sediment transport rather than bedload.

The tendency for a to decrease with increasing stream size, however, requires no geomorphic interpretation. It is simply a mathematical consequence of using a power function to model the rating relation. A generic power function equivalent to equation (1) can be written:

$$Y = aX^{b} \qquad (3)$$

Here, the substitution of X and Y for Q and Q_c is intended to emphasize the fact that the mathematical properties of the equation are independent of any geomorphic considerations. Solving for a yields:

$$a = Y/X^{b} \qquad (4)$$

It is clear from equation (4) that a is inversely proportional to X^b, such that, for constant Y, a must decrease whenever X increases, regardless of what X represents. It can also be seen that the magnitude of the decrease in a for a given change is X is greatly magnified by an increase in b, provided X is greater than unity. In general, any increase in b will cause a disproportionately large change in a, and will amplify the effect of increasing or decreasing X.

CONCLUSIONS

Sediment rating curves are commonly developed by fitting transport data to a simple 2-parameter power function that implicitly incorporates an assumption that sediment transport persists at all discharge levels. This assumption may be valid for the transport of fine suspended sediment, but is an unrealistic representation of coarse bedload transport, which generally ceases at discharges below a non-zero threshold (Q_c). The existence of such a threshold can be accommodated by fitting the transport data to a 3-parameter power function in which an entrainment threshold is subtracted from the measured water discharge.

Numerical experiments demonstrate that inappropriately fitting sediment transport data with a 2-parameter power function can result in large errors in both parameter estimates. Fitting the 2-parameters model to synthetic data with discharges ranging from Q_c to $2Q_c$ produced estimates of the exponent, b, between 4 and 5 time larger than the correct value, and estimates of the coefficient, a, that are orders of magnitude too small. Application of the resulting rating curves to a hypothetical hydrograph showed that these parameter errors can propagate to large errors in estimated sediment loads, particularly if extrapolation beyond the range of the measured data is required. In a similar analysis of bedload transport and stream flow data collected in a California stream, the value of b determined with a 2-parameter rating curve exceeded the value of b determined with a 3-parameter transport model by 42 percent. The estimated annual load based on the 2-parameter model and daily stream flow data is 80 percent larger than the annual load estimated with the 3-parameter model.

In general, forcing Q_c to zero by fitting bedload data with a 2-parameter power function exaggerates the value of the b. The degree to which b is exaggerated increases as the ratio of Q_c to the range of discharges sampled increases. The value of a scales in proportion to $1/Q^b$. Consequently, errors in b can lead to disproportionately large errors in a, depending on the range of discharges in the sample data. Overall, distortion of these parameter values caused by

improperly forcing Q_c to zero undermine efforts to interpret the physical meaning of variability in rating curve parameter values. Statistical relationships between rating curve parameters and various physical attributes of streams described in the literature may actually reflect spurious correlations related to the selection of physical metrics that co-vary with Q_c or to mathematical properties of the power function model.

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