TWO-DIMENSIONAL POISSONIAN HOMOGENEOUS MODEL FOR SUSPENDED SEDIMENT AND POLLUTANT MOVEMENTS IN OPEN-CHANNEL FLOW

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Abstract: When classic equations are used to study the sediment movements in open-channel flows, some problems appear, relating to: (i) the non-linearity of the equations, (ii) the complexity of the liquid and solid interactions and (iii) the unawareness of the liquid and solid movement's changes. The Random Theory avoids such problems and proposes a kinematic analysis of the flow and solid particles movements, exploiting, simultaneously, the open flux turbulent characteristics. The two-dimensional suspended sediment or pollutant trajectories $\omega(x,z,t)$ result from the combination of two 1-D chronological displacements series in the $i = 1, 3$ senses, intercalated with periods of time when the grain does not move in these senses. These two series are interdependent. When the one-dimensional series are described by Homogeneous Poissonian Random Process, the resultant two-dimensional model is also Homogeneous Poissonian, defined by four mobility density functions, which characterize the particles' movement. The objectives of this work are: (i) to present the longitudinal and vertical two-dimensional Random Process Model; (ii) to calibrate and validate it, using radiotracers data from laboratory channels experiments; (iii) to show that the Random Theory may be applied to evaluate the 2-D sediment and pollutant movements in open-channel flows, for instantaneous and continuous injections conditions. To validate the model, data obtained from radioactive applications in a prismatic channel 12.0 m long, 0.40 m width and 0.60 m height of the *Central Hydraulic Laboratory of France – LCHF* were used. They have showed that the 2-D Random Process may describe, with precision, the suspended movement of fine sediment particles and/or contaminants in open channel flows, whatever kind of immersion may be.

INTRODUCTION

The bed and suspended load movements of sediments and contaminant particles in open channel flows characterize stochastic process, where the elementary events are the single grains' trajectories. They are dependent of the liquid phase turbulent structure, or in other words, of the hydrodynamic process. The trajectories or achievements of the single particle or of the group of particles can be analyzed by Lagrangean or Spatial and Eulerian or Temporal Descriptions and by the Random Processes Theory. Two stochastic processes are considered:

$$\vec{R}(t,\omega) = \left[X(t,\omega), Y(t,\omega), Z(t,\omega)\right] = X_{ti}(\omega);\ i=1,2,3 \tag{1}$$

that characterizes the evolution of the particle's position vector as a function of time, which longitudinal, lateral and vertical components are $X(t,\omega)$, $Y(t,\omega)$ and $Z(t,\omega)$, respectively. The second 3D stochastic process:

$$T(x,y,z,\omega) = \left[T(x,\omega), T(y,\omega), T(z,\omega)\right] = T_{xi}(\omega);\ i=1,2,3 \tag{2}$$

characterizes the particle's passing time by the point of coordinates (x,y,z).

$T(x,\omega)$, $T(y,\omega)$ and $T(z,\omega)$ represent the times spent by the particle to travel the distances $0x$, $0y$ and $0z$, respectively. ω represents the trajectory or the sediment particle achievements, as presented in Figure 1, following. $X_{ti}(\omega)$ and $T_{xi}(\omega)$ processes can be defined by their Probability Distribution Functions:

$$F_t(x_i) = P\{X_i(t,\omega) \le x_i\};\ x_i \ge 0;\ i=1,2,3 \tag{3}$$

$$Q_{xi}(t) = P\{T(x_i,\omega) \le t\};\ t \ge 0;\ i=1,2,3 \tag{4}$$

which are related to each other by Todorovic´'s Equation (5) (Todorovic´ *et al.*, 1966; Wilson-Jr., 1987):

$$F_t(x_i) = 1 - Q_{xi}(t); \quad i=1,2,3 \tag{5}$$

They had shown that the Probability Distribution Function of these random processes can be expressed in terms of two pairs of Approximate Functions $F_{t1}(x_i)$ and $F_{t2}(x_i)$; $Q_{x1i}(t)$ and $Q_{x2i}(t)$; $i = 1,2,3$, respectively, such that:

$$0 \le F_{t1i}(x_i) \le F_{ti}(x_i) \le F_{t2i}(x_i) \le 1; \quad i=1,2,3 \tag{6}$$

$$0 \le Q_{x2i}(t) \le Q_{xi}(t) \le Q_{x1i}(t) \le 1; \quad i=1,2,3 \tag{7}$$

In each direction, e. g. in the longitudinal direction $0x_{i=1}$, where $x_i = x_1 = x$, the Approximate Distribution Functions $F_{tj}(x)$ and $Q_{xj}(t)$, $j = 1,2$ can be explained as functions of two new stochastic processes $G_n^{0,x}$ and $E_n^{0,t}$ from the same elementary events ω:

$$G_n^{0,x} = \{\mu_{0,x} = n\} \tag{8}$$

which represents the medium number of grain displacements, $\mu_{0,x}$ over the distance $[0, x]$, and,

$$E_n^{0,t} = \{\eta_{0,t} = n\} \tag{9}$$

the medium number of grain displacements, $\eta_{0,t}$ over the time period $[0, t]$.

$G_n^{0,x}$ and $E_n^{0,t}$ are Markovian Processes with similar properties. So, for the set $G_n^{0,x}$, it has:

$$\left\{\begin{array}{l} P\{G_1^{x,x+\Delta x} | G_k^{0,x}\} = \lambda_2(x,k)\Delta x + \vartheta(\Delta x) \\ P\{G_\nu^{x,x+\Delta x} | G_k^{0,x}\} = \vartheta(\Delta x) \quad \nu \ge 2 \\ P\{G_0^{x,x+\Delta x} | G_k^{0,x}\} = 1 - \lambda_2(x,k)\Delta x + \vartheta(\Delta x) \\ P\{G_0^{0,0}\} = 1 \end{array}\right\} \qquad \Delta x \to 0 \tag{10}$$

where $\vartheta(\Delta x)$ is a grain first order infinitesimal displacement distance.

The $G_n^{0,x}$ and $E_n^{0,t}$ occurrence probabilities are solutions of the system of equations derived from these properties:

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial x} P\{G_k^{0,x}\} = \lambda_2(x,k-1) P\{G_{k-1}^{0,x}\} - \lambda_2(x,k) P\{G_k^{0,x}\} \\ \dfrac{\partial}{\partial x} P\{G_0^{0,x}\} = -\lambda_2(x,0) P\{G_0^{0,x}\} \end{array}\right. \tag{11}$$

with the following initial conditions:

$$x = 0 \begin{cases} P\{G_0^{0,x}\} = 1 \\ P\{G_k^{0,x}\} = 0; \ k \ge 1 \end{cases} \tag{12}$$

Similar analytical expressions to the Equations (10), (11) and (12) are obtained for the $E_n^{0,t}$ process. The solution of these differential equations yields the probability laws for the numbers of displacements in time and spatial intervals.

Two functions $\lambda_1(t,n)$ and $\lambda_2(x,n)$ appear, which describe the sediment particle mobility, in time and in that particular direction $x_{i=1} = x_1 \approx x$. Considering the three directions of the orthogonal axes $0xi$, $i = 1, 2, 3$, three pairs of Mobility Functions $\lambda_{1i}(t,n)$ and $\lambda_{2i}(x_i,n)$ are obtained, which describe the sediment grains 3D movements, in time and space. In each x_i direction it has been:

$$\begin{cases} \lambda_{1i}(t,n_i) = \lim_{\Delta t \to 0} \dfrac{P\left\{E_{1i}^{t,t+\Delta t} \middle| E_{n_i}^{0,t}\right\}}{\Delta t} \\ \lambda_{2i}(x_i,n_i) = \lim_{\Delta x_i \to 0} \dfrac{P\left\{G_{1i}^{x_i,x_i+\Delta x_i} \middle| G_{n_i}^{0,x_i}\right\}}{\Delta t} \end{cases} \qquad i = 1,2,3 \tag{13}$$

PARTICLES' RANDOM TRAJECTORIES

The particle's trajectory ω results from the combination of three chronological displacement series: (i) an alternate series of longitudinal displacements in the direction of the flow, with intercalated periods of time when the particle ceases to progress in this sense; (ii) an alternate series of vertical displacements in the direction of gravity , with intercalated periods when the particle ceases to decant; (iii) a series of lateral displacements, with intercalated periods when the particle ceases to move laterally, as shown in Figures 1 and 2 (Wilson-Jr., 2004). One can observe that: (1) when the particle ceases to progress in the direction of flow, it can be decanting or moving laterally, (2) when it ceases to decant, it can be moving in the longitudinal or lateral directions, (3) when it ceases to progress towards one of the walls, it can be decanting or moving in the longitudinal direction. In this manner, the three series are interdependent and should be considered together.

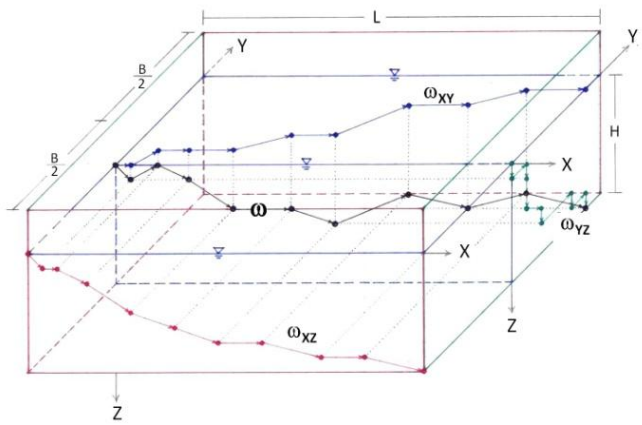

Figure 1. Sediment particle trajectory and its components in open channel flow (Wilson-Jr., 2004)

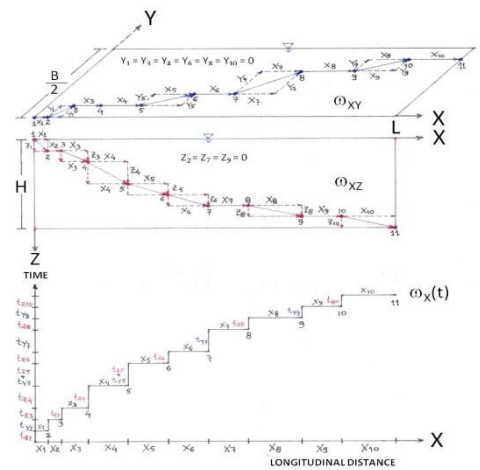

Figure 2. Particle trajectories on the horizontal ω_{XY} and vertical ω_{XZ} plans and longitudinal chronological series ω_X (t) (Wilson-Jr., 2004)

As positive longitudinal and vertical displacements are predominant these random series are more appropriate to describe the particle movements than the series of lateral displacements, where the negative movement (opposite to the direction of one of the walls) no can be neglected. Thus, the 2D resulting Random Processes that characterize the Lagrangean and Eulerian Descriptions, become respectively equal to:

$$\vec{R}(t,\omega) = \left[X(t,\omega), Z(t,\omega)\right] = X_{ti}(\omega); \qquad i=1,3 \tag{14}$$

$$T(x,z,\omega) = \left[T(x,\omega), T(z,\omega)\right] = T_{xi}(\omega); \qquad i=1,3 \tag{15}$$

OBJECTIVES

General Objectives

The general purpose of this work includes the study of Lagrangean and Eulerian Descriptions of bed and in suspension sediments and contaminants particles' movements and cases of instantaneous and continuous immersion of these particles in open channel flows, using the Theory of Random Process.

Specific Objectives

In particular, it is intended: (i) show that the Theory of Random Process accurately describe the 2D movements,

longitudinal and vertical, of sediment and pollutants in open channel flows; (ii) present the 2-D Poissonian models of sediments and contaminants in longitudinal and vertical suspension movements; (iii) highlight the importance of Temporal and Spatial Intensity Mobility Functions in the definition of random models, as well as in calibration and validation of 2D models, with data obtained in laboratory channels and nature.

3-D LAGRANGEAN RANDOM PROCESS

To better illustrate the ω trajectories of the particles, it will adopt Lagrangean Description of the Random Processes $X_{ti}(\omega)$, $i = 1,2,3$; shown in Figure 2.

Probability Density Function $f_t(x, y, z)$

Conceding the mutual independence of the Random Processes $X_{ti}(\omega)$, $i = 1,2,3$; the Probability Distribution Function of the particles' position with respect to time can be simplified, and be described by the general equation (Sayre and Conover, 1967; Wilson-Jr., 1987, 2004):

$$F_t(x,y,z) = \sum_{n=0}^{\infty} P\left\{ \sum_{j=0}^{N(t)} X_j \le x,\ \sum_{j=0}^{N(t)} Y_j \le y,\ \sum_{j=0}^{N(t)} Z_j \le z,\ N(t) = n \right\} \tag{16}$$

where $N(t)$ represents the number of particle displacements in time interval $[0,t]$. Thus, the Probability Density Function of the position of the particle in time t, is given by:

$$f_t(x,y,z) = \frac{\partial^3 F_t(x,y,z)}{\partial x \partial y \partial z} = f_t(x) f_t(y) f_t(z) P\left\{ \sum_{n=0}^{\infty} N(t) = n \right\} = f_t(x) f_t(y) f_t(z) \tag{17}$$

The pairs of Mobility Functions $\lambda_{1i}(t,n)$ and $\lambda_{2i}(x_i,n)$, $i = 1,2,3$; also called Particle Kinematic Change Functions, characterize the particles temporal and spatial movements, respectively (Wilson-Jr., 1987; Monteiro, 2004; Wilson-Jr. and Monteiro, 2013). When the probability of the grains' displacements, in time and distance intervals $[t,\ t+\Delta t]$ and $[x,\ x+\Delta x]$, Δt and Δx tending to zero, are independents of time, particle position and previous displacements, i.e., independents of the sediment particle history, the particle movement is called <u>*out of memory*</u>. In this case, the Mobility Functions are positive constants and the Density Probability Functions $f_t(x_i)$, $i = 1, 2, 3$ are described by Homogeneous Poissonian Random Processes (Wilson-Jr., 1987). For the 3D model results:

$$0.0 \le f_{t1}(x_i) \le f_t(x_i) \le f_{t2}(x_i) \le 1.0 \tag{18}$$

$$\begin{cases} f_{t1}(x_i) = \lambda_{2i}\, e^{-\lambda_{1i} t - \lambda_{2i} x_i} \displaystyle\sum_{k=0}^{\infty} \frac{(\lambda_{1i} t)^k}{k!} \frac{(\lambda_{2i} x)^k}{k!} \\ f_{t2}(x_i) = \lambda_{2i}\, e^{-\lambda_{1i} t - \lambda_{2i} x_i} \displaystyle\sum_{k=0}^{\infty} \frac{(\lambda_{1i} t)^{k+1}}{(k+1)!} \frac{(\lambda_{2i} x)^k}{k!} \end{cases} \qquad i = 1, 2, 3 \tag{19}$$

For any kind of sediment and/or pollutant immersion, if there is no loss or gain of particles in the course of time and distance, the total volume occupied by the particles is preserved after immersion time t_d. That is, it has been:

$$M_t = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \int_{z=0}^{\infty} f_t(x,y,z)\, dx\, dy\, dz = 1; \quad t \ge t_d \ge 0 \tag{20}$$

To illustrate the performance of the 3D model components, plots of the probability density functions for some values of the mobility functions λ_{1i} and λ_{2i}, $i = 1,2,3$, are presented in Figures 3, 4 and 5 (Wilson-Jr., 2004). The mobility values are obtained by comparing the experimental and theoretical variations of the particles' position moments as function of time, and/or of the particles' passing time moments through cross sections. T^{-1} and L^{-1} in the figures correspond to the units of λ_{1i} and λ_{2i}, $i = 1,2,3$, that is, the inverse of time and length units, respectively.

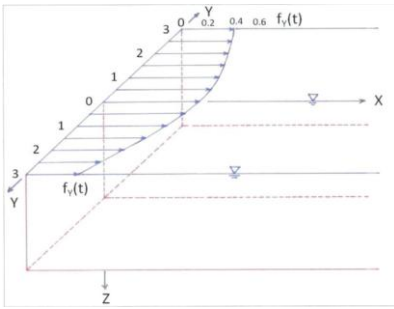

Figure 3. Probability density function of lateral displacements in time. ($\lambda_{1y} = 0.07T^{-1}$; $\lambda_{2y} = 0.90L^{-1}$)

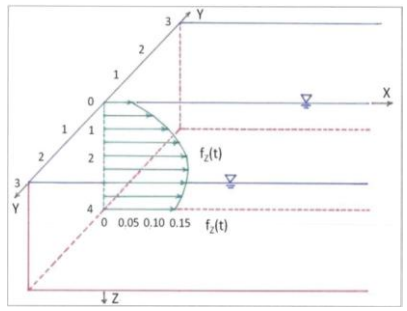

Figure 4. Probability density function of vertical displacements in time. ($\lambda_{1z} = 0.15T^{-1}$; $\lambda_{2z} = 1.15L^{-1}$)

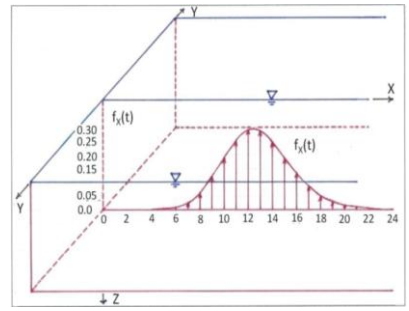

Figure 5. Probability density function of longitudinal displacements in time. ($\lambda_{1x} = 0.15T^{-1}$; $\lambda_{2x} = 3.50L^{-1}$)

To determine Probability Density Functions $f_t(x)$, $f_t(y)$ and $f_t(z)$, the software PAICON (***P**rocessos **A**leatórios com Injeção **I**nstantânea e **CON**tínua*) was used, implemented in *Delphi language* from *Inprise Corporation* (Monteiro and Wilson-Jr., 2002; Monteiro, 2004). These curves are fitted to the experimental results of projects on sediment and pollutants' transport and dispersion with the use of tracers: radioactive, fluorescents or chemicals, among others. In the Figure 6 is shown the points of the Rhodamine-B transit curve through a River *Loire*'s cross section and the upper and lower theoretical probability density approximate curves $q_{xj}(t), j = 1,2,$ adjusted with the use of the PAICON Program (Monteiro and Wilson-Jr., 2003). The tracer application technique known as Double Labelling Method (*Méthode de Double Marquage*) was used. Simultaneous labelings and injections of water and fine sediments (special type named *schlam*) with Rhodamine-B and Au^{198}, respectively, allowed a comparative study of sediment and fluid transfer properties, in the segment between *Grangent* and *Villerest* Dams, on the River *Loire* (Tola *et al*., 1981; Wilson-Jr., 1987).

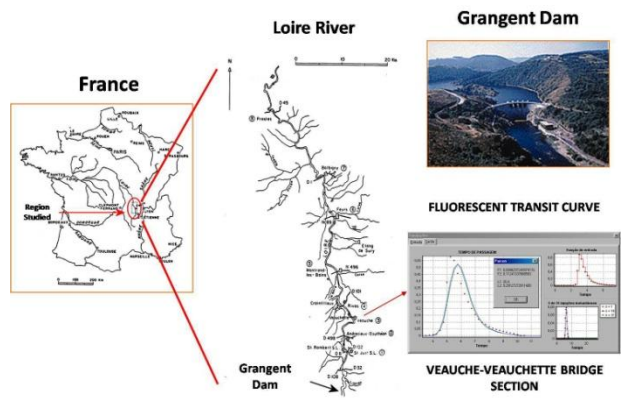

Figure 6. Rhodamine-B transit curve through the *Veauche-Veauchette* Bridge section, on the River *Loire* (Tola *et al*., 1981; Monteiro and Wilson-Jr., 2003)

Intensity of Particle Mobility Functions

The Mobility Functions λ_{1i} and λ_{2i}, $i = 1,2,3$, defined by Equations (13) appeared from the analytical development of Random Processes $X_{ti}(\omega)$ and $T_{xi}(\omega)$ to explain sediment mobility in each direction i and on the instant t. Different models can be obtained from the mathematical expressions that define these mobility functions, which should consider the sediment and/or contaminant particle characteristics, as well the hydrodynamic properties. The general expressions for λ_{1i} and λ_{2i}, $i = 1,2,3$, obtained by Vukmirovic´(1975) and Wilson-Jr. (1987, 2012) consider the mobility of the particles as a function of time, of the distances traveled in one direction and of its past performance in time (n) and distance (k), in each direction (i):

$$\begin{cases} \lambda_{1i}(t,n) = \lambda_{1i}(t)\,\lambda_{2i}(n) \\ \lambda_{2i}(x_i,k) = \lambda_{2i}(x_i)\,\lambda_{2i}(k) \end{cases} \qquad i = 1, 2, 3 \qquad (21)$$

$$\begin{cases} \lambda_{1i}(t,n) = \lambda_{1i} = const. \\ \lambda_{2i}(x_i,k) = \lambda_{2i} = const. \end{cases} \qquad i = 1, 2, 3 \qquad (22)$$

where: λ_{1i} is the particle mobility factor in a given time t in the direction i; and
λ_{2i} the particle mobility factor in a certain position x_i in the direction i.

2-D HOMOGENEOUS LAGRANGEAN RANDOM POISSONIAN PROCESSES

For these cases the grain mobility functions in the longitudinal and vertical directions assume constant values in accordance with Equation (22), or simply:

$$\text{for } i = 1 = x \Rightarrow \begin{cases} \lambda_{11} = \lambda_{1x} = const_{11} \\ \lambda_{21} = \lambda_{2x} = const_{21} \end{cases} \tag{23}$$

$$\text{for } i = 3 = z \Rightarrow \begin{cases} \lambda_{13} = \lambda_{1z} = const_{13} \\ \lambda_{23} = \lambda_{2z} = const_{23} \end{cases} \tag{24}$$

Some of the main statistical properties that characterize the 2-D Homogeneous Lagrangean Random Process $X_t(x,z)$, for cases of **instantaneous** and **continuous immersions**, are analytically described by the following equations (Monteiro, 2004; Wilson-Jr. and Monteiro, 2004):

Instantaneous Immersion Case

Probability density function $f_t(x,z)$

$$0.0 \le f_{t1}(x,z) \le f_t(x,z) \le f_{t2}(x,z) \le 1.0 \tag{25}$$

$$\begin{cases} f_{t1}(x,z) = \lambda_{21}\,\lambda_{23}\,e^{-\lambda_{11}t-\lambda_{21}x}\,e^{-\lambda_{13}t-\lambda_{23}z}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda_{11}t)^n}{n!}\frac{(\lambda_{21}x)^n}{n!}\frac{(\lambda_{13}t)^k}{k!}\frac{(\lambda_{23}z)^k}{k!} \\ f_{t2}(x,z) = \lambda_{21}\,\lambda_{23}\,e^{-\lambda_{11}t-\lambda_{21}x}\,e^{-\lambda_{13}t-\lambda_{23}z}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda_{11}t)^{n+1}}{(n+1)!}\frac{(\lambda_{21}x)^n}{n!}\frac{(\lambda_{13}t)^{k+1}}{(k+1)!}\frac{(\lambda_{23}z)^k}{k!} \end{cases} \tag{26}$$

Probability distribution function $F_t(x,z)$

$$0.0 \le F_{t1}(x,z) \le F_t(x,z) \le F_{t2}(x,z) \le 1.0 \tag{27}$$

$$\begin{cases} F_{t1}(x,z) = e^{-\lambda_{11}t-\lambda_{21}x}\,e^{-\lambda_{13}t-\lambda_{23}z}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(\lambda_{11}t)^n}{n!}\frac{(\lambda_{21}x)^k}{k!}\sum_{m=0}^{\infty}\sum_{p=m}^{\infty}\frac{(\lambda_{13}t)^m}{m!}\frac{(\lambda_{23}z)^p}{p!} \\ F_{t2}(x,z) = e^{-\lambda_{11}t-\lambda_{21}x}\,e^{-\lambda_{13}t-\lambda_{23}z}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(\lambda_{11}t)^{n+1}}{(n+1)!}\frac{(\lambda_{21}x)^k}{k!}\sum_{m=0}^{\infty}\sum_{p=m}^{\infty}\frac{(\lambda_{13}t)^{m+1}}{(m+1)!}\frac{(\lambda_{23}z)^p}{p!} \end{cases} \tag{28}$$

Median position of particles $M_t(x,z)$

$$M_{t1}(x,z)_{Inst.} \le M_t(x,z)_{Inst.} \le M_{t2}(x,z)_{Inst.} \tag{29}$$

$$\begin{cases} M_{xz1}(t)_{Inst.} = E\left[F_{t1}(x,z)\right] = \frac{1}{\lambda_{21}\,\lambda_{23}}\left[\lambda_{11}\,\lambda_{13}\,t^2 + (\lambda_{11}+\lambda_{13})t + 1\right] \\ M_{xz2}(t)_{Inst.} = E\left[F_{t2}(x,z)\right] = \frac{\lambda_{11}\,\lambda_{13}}{\lambda_{21}\,\lambda_{23}}\,t \end{cases} \tag{30}$$

Variance of the particles' positions $\left[S^2_{xzj}(t)\right]_{Inst.}$

$$\left[S^2_{xz1}(t)\right]_{Inst.} \le \left[S^2_{xz}(t)\right]_{Inst.} \le \left[S^2_{xz2}(t)\right]_{Inst.} \tag{31}$$

$$\begin{cases} \left[S_{xz1}^2(t)\right]_{Inst.} = \dfrac{2\lambda_{11}\lambda_{13}(\lambda_{11}+\lambda_{13})t^3 + \left[(\lambda_{11}+\lambda_{13})^2 + 10\,\lambda_{11}\lambda_{13}\right]t^2 + 6(\lambda_{11}+\lambda_{13})t + 3}{(\lambda_{21})^2(\lambda_{23})^2} \\ \left[S_{xz2}^2(t)\right]_{Inst.} = \dfrac{2\lambda_{11}\lambda_{13}(\lambda_{11}+\lambda_{13})t^3 + 2\lambda_{11}\lambda_{13}t^2}{(\lambda_{21})^2(\lambda_{23})^2} \end{cases} \tag{32}$$

Continuous Immersion Case

This case consists of a uniform injection, laterally distributed in the free surface at the channel's upstream extremity during a period of time t_d, as illustrated in Figures 7 and 8. In the experiments conducted at the LCHF (*Laboratoire Central d'Hydraulique de France*), t_d varied from 7.0 to 13.0 minutes (Wilson-Jr., 1987).

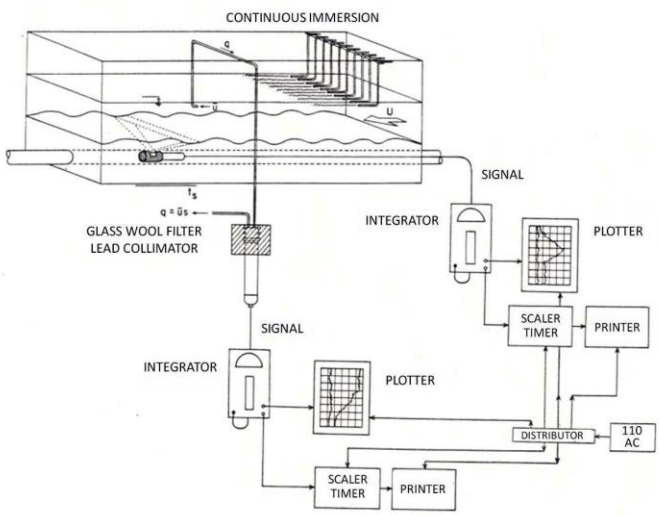

Figure 7. Immersion and detection system of bed and suspended movements of fine sediments in laboratory channel (Wilson-Jr., 1987)

Figure 8. Continuous immersion of fine sediments on the free surface of LCHF's channel

In the following equations $t_o \geq t_d$ represents the period of time of continuous recording of tracer's passage by the sampling points. To shorten the text, only the statistical properties related to the upper approximate probability density function of the 2-D Homogeneous Lagrangean Random Processes $X_{ti}(\omega)$, $i = 1,3$ will be presented.

Probability density function $h_{to}(x,z)$

$$h_{to}(x,z) = \frac{1}{t_d}\int_0^{t_d} f_{t_o-\tau}(x)\, f_{t_o-\tau}(z)\, d\tau \tag{33}$$

For the 2-D Poissonian Random Process, the upper approximate probability density function $h_{to2}(x,z)$ becomes:

$$\begin{aligned} h_{to2}(x,z) = \frac{1}{t_d}\int_0^{t_d} & \lambda_{21}\, e^{-\lambda_{21}x}\, e^{-\lambda_{11}(t_o-\tau)} \sum_{n=0}^{\infty}\left\{\frac{\left[\lambda_{11}(t_o-\tau)\right]^{n+1}}{(n+1)!}\frac{(\lambda_{21}x)^n}{n!}\right\}\times \\ & \lambda_{23}\, e^{-\lambda_{23}z}\, e^{-\lambda_{13}(t_o-\tau)} \sum_{n=0}^{\infty}\left\{\frac{\left[\lambda_{13}(t_o-\tau)\right]^{n+1}}{(n+1)!}\frac{(\lambda_{23}x)^n}{n!}\right\} d\tau \end{aligned} \tag{34}$$

Probability distribution function $H_{to}(x,z)$

$$H_{to}(x,z) = \frac{1}{t_d}\int_0^{t_d} F_{t_o-\tau}(x)\, F_{t_o-\tau}(z)\, d\tau \tag{35}$$

The distribution functions $F_{to\text{-}\tau}(x)$ and $F_{to\text{-}\tau}(z)$ are obtained from the 1-D models approximate equations in the longitudinal and vertical directions, respectively, that is, from the 1-D Homogeneous Poissonian equations system, in the $i = 1, 3$ directions, with instantaneous immersions (Monteiro, 2004):

$$0.0 \le F_{t1}(x_i) \le F_t(x_i) \le F_{t2}(x_i) \le 1.0 \tag{36}$$

$$\begin{cases} F_{t1}(x_i) = e^{-\lambda_{1i}\, t - \lambda_{2i}\, x_i} \sum_{\nu=0}^{\infty} \sum_{k=\nu+1}^{\infty} \dfrac{(\lambda_{1i}\, t)^{\nu}}{\nu!} \dfrac{(\lambda_{2i}\, x_i)^{k}}{k!} \\ F_{t2}(x_i) = e^{-\lambda_{1i}\, t - \lambda_{2i}\, x_i} \sum_{\nu=0}^{\infty} \sum_{k=\nu}^{\infty} \dfrac{(\lambda_{1i}\, t)^{\nu}}{\nu!} \dfrac{(\lambda_{2i}\, x_i)^{k}}{k!} \end{cases} \tag{37}$$

Thus, for the Upper Approximate Probability Distribution Function $F_{(to\text{-}\tau)2}(x_i)$, $i = 1,3$, it has been:

$$F_{(t_o-\tau)2}(x_i) = e^{-\lambda_{1i}(t_o-\tau) - \lambda_{2i}\, x_i} \sum_{\nu=0}^{\infty} \sum_{k=\nu}^{\infty} \frac{[\lambda_{1i}(t_o-\tau)]^{\nu}}{\nu!} \frac{(\lambda_{2i}\, x_i)^{k}}{k!} \qquad i = 1,3 \tag{38}$$

Average position of the particles

The average position of the particles is given by the first-order moment, defined by:

$$\left[M^1_{xz}(t_o)\right]_{Cont.} = \frac{1}{t_d} \int_0^{t_d} M^1_x(t_o-\tau)\, M^1_z(t_o-\tau)\, d\tau \tag{39}$$

which can be estimated by two approximate first order moments, such that:

$$\left[M^1_{xz1}(t_o)\right]_{Cont.} \le \left[M^1_{xz}(t_o)\right]_{Cont.} \le \left[M^1_{xz2}(t_o)\right]_{Cont.} \tag{40}$$

Combining Equations 28, 29, 30, 39 and 40, one obtains the approximate expressions of the average position of the particles. The upper approximation is given by:

$$\left[M^1_{xz2}(t_o)\right]_{Cont.} = \frac{\lambda_{11}\, \lambda_{13}}{\lambda_{21}\, \lambda_{23}} \left(t_o^2 - t_o\, t_d + \frac{t_d^2}{3} \right) \tag{41}$$

Variance of the particles' position

Similarly, one can obtain the approximate expressions of the second order moment and of the Variance from the following definitions:

$$\left[M^2_{xz}(t_o)\right]_{Cont.} = \frac{1}{t_d} \int_0^{t_d} M^2_x(t_o-\tau)\, M^2_z(t_o-\tau)\, d\tau \tag{42}$$

$$\left[S^2_{xz}(t_o)\right]_{Cont.} = \left[M^2_{xz}(t_o)\right]_{Cont.} - \left[M^1_{xz}(t_o)\right]^2_{Cont.} \tag{43}$$

SUSPENDED SEDIMENT MOVEMENTS IN LABORATORY OPEN CHANNEL

An original experimental device was developed for the study of bed and suspended load with vertical transfers in a LCHF's channel 12.0 m long having a rectangular cross section 0.40 m wide by 0.60 m deep (Wilson-Jr., 1987). The lateral walls are made of glass, allowing for visual observations of bed configurations evolutions and the following of injections of sediments labeled with tracers. The channel was adapted, as schematized in Figure 7, such that hydraulic and sedimentologic measurements could be performed simultaneously with conventional methods.

Among several hydraulic and sedimentologic measurements performed, the following are highlighted: (i) the 3-D liquid velocities field; (ii) transport and dispersion of bed load particles; (iii) the vertical and transversal concentration profiles of suspended sediments continuously injected in the open surface of the channel, during time intervals; and (iv) the temporal and specific evolutions of the sediments deposited in the channel bed.

Examples of vertical profiles of cohesive and non-cohesive sediment concentration obtained in cross sections of the channel are presented in Figures 9 and 10, while examples of similar theoretical curves obtained from the probability density functions $f_t(x,z)$ are shown in Figure 11.

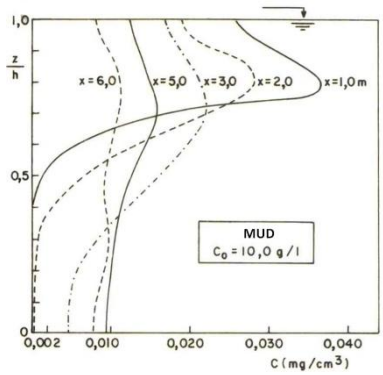

Figure 9. Vertical profiles of suspended cohesive sediment concentration across fixed section (Wilson-Jr., 1987)

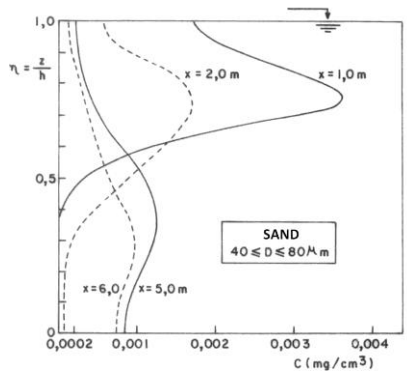

Figure 10. Vertical profiles of suspended non cohesive sediment concentration across fixed section (Wilson-Jr., 1987)

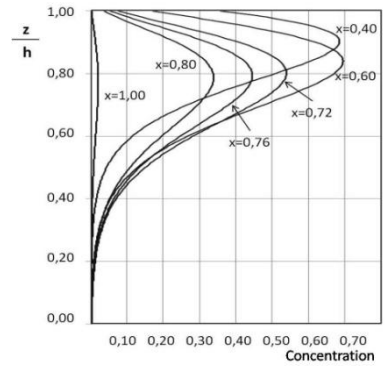

Figure 11. Theoretical vertical profiles of suspended cohesive sediment concentration across fixed section.

The comparison of these theoretical and experimental curves indicate that the 2D Random Models are so promising as the 1D models that the authors have been applied in laboratory channels and nature, in projects and investigations of Civil and Environmental Engineering.

SUSPENDED SEDIMENT MOVEMENTS RESULTS

Mendes and Wilson-Jr. (1998), Wilson-Jr. and Monteiro (2004) showed that the Homogeneous One-dimensional Poissonian Models described precisely the sediment and pollutant passage time through open channel cross sections. Data from experiments performed in Brazil and France consisting on the instantaneous immersion of tracers in suspension an upstream section and on the determination of the transit time's curves through some downstream cross sections were used for the 1-D applications using the PAICON-1D software. Some of these applications are highlighted in Wilson-Jr. and Monteiro (2004, 2013).

For the implementation of 2-D Poissonian models, the PAICON program was restructured to include the two-dimensional case and modern techniques of graphic outputs. This PAICON-2D version calculates the values of the approximate probability density and distribution functions of the Lagrangean and Eulerian descriptions, to estimated or experimentally obtained values of the λ_{11}; λ_{13}; λ_{21} and λ_{23} Mobility Functions.

To calibrate and validate the 2D Random Models is necessary to determine the values of these functions in the directions: longitudinal (λ_{11} and λ_{21}) and vertical (λ_{13} and λ_{23}). They are determined by the probability density functions moments of first and second order in each direction and by the longitudinal and vertical mean velocity of the particles, namely by Equations (44) to (47), following (Wilson-Jr. and Monteiro, 2013).

Temporal and spatial mobility of the particles in the x_i; i=1,3 directions

$$\begin{cases} \lambda_{1x_i} = \lim_{\Delta t \to 0} \dfrac{P\left\{E_{1x_i}^{t,t+\Delta t}\right\}}{\Delta t} \\ \lambda_{2x_i} = \lim_{\Delta x_i \to 0} \dfrac{P\left\{G_{1x}^{x_i,x_i+\Delta x_i}\right\}}{\Delta x_i} \end{cases} \qquad i = 1,3 \tag{44}$$

Longitudinal and vertical mean positions of a group of particles

$$\begin{cases} X_m(t) = \dfrac{\lambda_{1x}}{\lambda_{2x}} t = \dfrac{\lambda_{11}}{\lambda_{21}} t \\ Z_m(t) = \dfrac{\lambda_{1z}}{\lambda_{2z}} t = \dfrac{\lambda_{13}}{\lambda_{23}} t \end{cases} \tag{45}$$

Variances of the longitudinal and vertical positions of a group of particles

$$S_j^2(t) = 2\frac{\lambda_{1j}}{\lambda_{2j}^2}t \qquad j = x, z \;\; ou \;\; j = 1, 3 \tag{46}$$

Longitudinal and vertical mean velocities of a group of particles

$$U_j = \frac{\lambda_{1j}}{\lambda_{2j}} \qquad j = x, z \;\; ou \;\; j = 1, 3 \tag{47}$$

Each pair of values of the Mobility Functions describes the sediment grain movement in one direction. The results of the evolution in time of their approximate superior probability density function are shown in Figures 12 and 13.

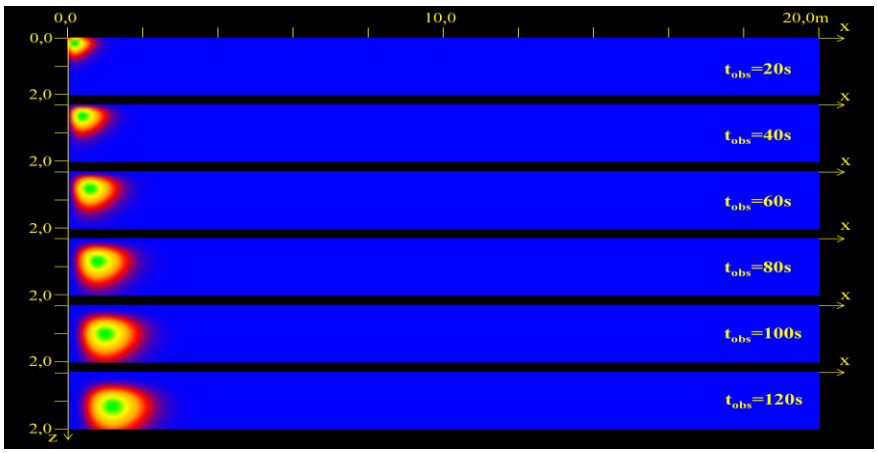

Figure 12. Cloud of sediments due to an instantaneous immersion at the free surface: case of equal longitudinal and vertical mobility.
($\lambda_{1x} = \lambda_{1z} = 0.10$ s^{-1}; $\lambda_{2x} = \lambda_{2z} = 10.0$ m^{-1}; $U_x = U_z = 0.01$ m s^{-1})

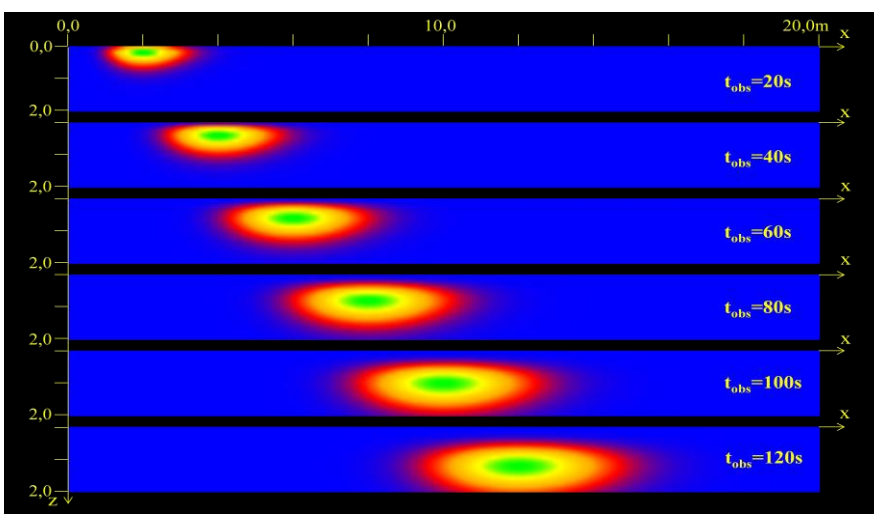

Figure 13. Cloud of sediments due to an instantaneous immersion at the free surface: in which case longitudinal mobility is greater than the vertical.
($\lambda_{1x} = 1.0$ s^{-1}; $\lambda_{1z} = 0.10$ s^{-1}; $\lambda_{2x} = \lambda_{2z} = 10.0$ m^{-1}; $U_x = 0.10$ m s^{-1}; $U_z = 0.01$ m s^{-1})

Their values are proportional to the concentration of sediment in suspension injected instantly to the free surface of a section located at the upstream extremity of the flow, for the cases where: (i) longitudinal and vertical mobility are equal; (ii) in the case of fine sediment which longitudinal mobility is about one order of magnitude higher than its vertical mobility or settling velocity. Similar studies were considered in the case of a continuous immersion for a time $[0, t_d]$ for t_d equal to 120 seconds. They are shown in Figures 14 and 15.

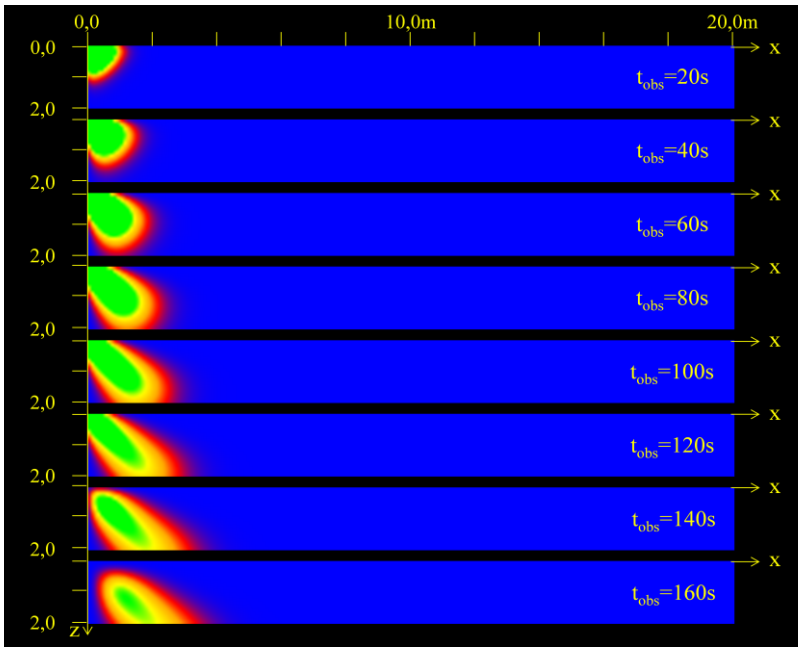

Figure 14. Plume of sediments due to continuous immersion at the free surface during the time interval [*0*, t_d]: case of equal longitudinal and vertical mobility. ($\lambda_{1x} = \lambda_{1z} = 0.20$ s^{-1}; $\lambda_{2x} = \lambda_{2z} = 10.0$ m^{-1}; $t_d = 120$ s; $U_x = U_z = 0.02$ m s^{-1})

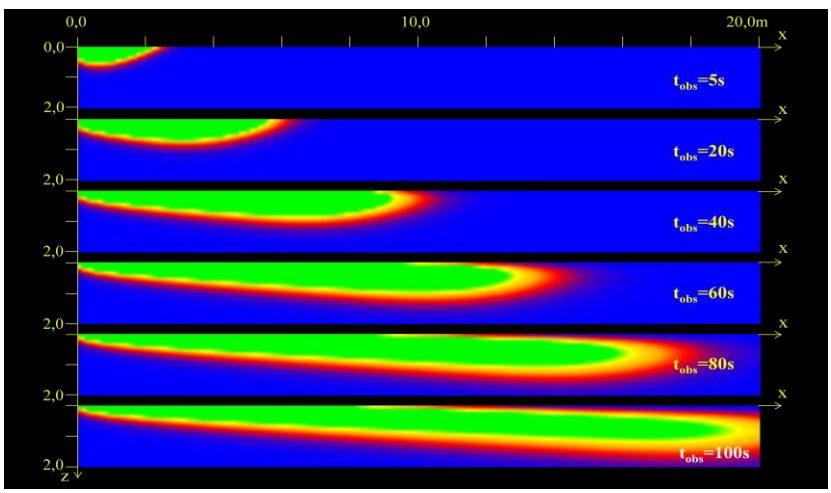

Figure 15. Plume of sediments due to continuous immersion at the free surface during the time interval [*0*, t_d]: in which longitudinal mobility is greater than the vertical. ($\lambda_{1x} = 2.0$ s^{-1}; $\lambda_{1z} = 0.10$ s^{-1}; $\lambda_{2x} = \lambda_{2z} = 10.0$ m^{-1}; $U_x = 0.20$, $U_z = 0.01$ m s^{-1})

CONCLUSIONS AND RECOMMENDATIONS

The Random Theory has shown that the trajectory ω of the particle results from the combination of two chronologic series of movement periods: (i) an alternate series of longitudinal downstream steps intercalated by time periods when the particle does not move in this sense, and (ii) an alternate series of fall vertical steps intercalated by time periods when the grain does not move vertically. These series are defined by the Mobility Functions: $\lambda_{x1}(t,n)$, $\lambda_{z1}(t,n)$, $\lambda_{t2}(x,n)$ and $\lambda_{t2}(z,n)$, which analytical expressions characterize the particles random movements.

It has been shown that the Random Process Theory is very enveloping, in such way that the diffusion dispersion Fickian classical equations of suspended sediment and pollutant movements are particular random process cases, characterized by constant values of the mobility functions, or in other words, by Homogeneous Poissonian Models.

When the sizes and concentrations of the sediment grains are reduced (fine sand, clays, and mud mixtures with concentrations lower than 150 mg/l), the solid particles behave as those of the fluid and the 1-D suspended movement can be described by 1-D Homogeneous Poissonian Models. This results is not surprising, since for large values of x and t, the Poissonian Model approaches the Gaussian Model, classical solution of the Diffusion-Dispersion equation.

When the grain mobility is not constant, more complexes models are generated, combining longitudinal and vertical non-homogeneous Poissonian and non-Poissonian models. Mobility Functions obtained in open channel flows, with grain of sediment labeled with radiotracers, permit the determination of the Mobility Functions and to adjust and validate the resultant random models.

Two research lines are opened: (i) the study of sediment and/or pollutant movement described by Mobility Functions that vary with time, distance and number of displacements performed in time and space, that is, performed by particles *with memory*; (ii) a description of the sediment movement by 2-D and 3-D processes. For these courses, a collection of data on sediment and pollutant movement obtained in laboratory channel and in nature with use of radioactive, dyes and chemical tracers is available for the use of this powerful Theory of Random Processes.

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